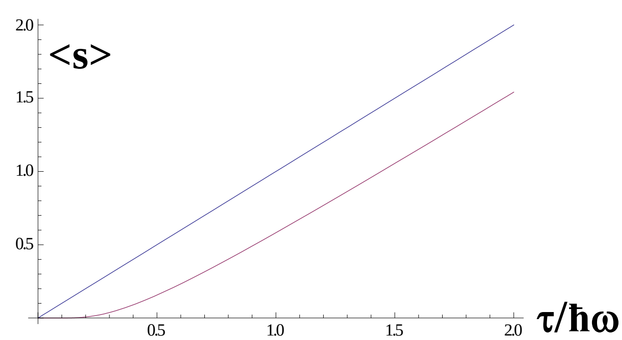
Lecture 10 Summary

Phys 404

We consider the spectrum of electromagnetic waves that emerge from a hollow box with walls at temperature τ. This problem has relevance to the cosmic microwave background spectrum, and the calibration of infrared thermometers.

Consider a single mode of a simple one-dimensional electromagnetic resonator. Imagine two parallel metal walls separated by a distance L. The fundamental mode of oscillation of the electromagnetic field will have a wavelength that is twice the distance between the plates to satisfy the boundary condition that the tangential electric field goes to zero at each wall. Hence $\lambda = 2L$, and the energy of this mode, according to Einstein is $E = \frac{hc}{\lambda} = \hbar\omega$, with $\omega = \frac{\pi c}{L}$ (This is where quantum mechanics sneaks into the argument). This particular mode can be occupied by either 0, 1, 2, 3, … photons, the quantum of electromagnetic energy. Einstein introduced the quantized energy value of the photon to explain the photoelectric effect. The higher the photon occupation number, the larger the amplitude of vibration of this particular mode. A classical electromagnetic wave would be recovered in the limit of large photon occupation number. The energy states of the mode now correspond to a ladder of energies $E = s\hbar\omega$, with $s = 0,1,2,3, \ldots$ This bears strong resemblance to the energy states of the quantum harmonic oscillator, except for the absence of the zero-point energy. Hence it is natural to consider the occupation of electromagnetic modes in terms of the energy states of a harmonic oscillator, as is done in the theory of quantum electrodynamics.

With this photon picture of the electromagnetic mode, we can now calculate the thermal average photon number in the mode. Start with the partition function $Z = \sum_s e^{-\varepsilon_s/\tau}$, where $\varepsilon_s = s\hbar\omega$, and $s = 0,1,2,3, \ldots$ The partition function was calculated in HW #3, problem 3 for a harmonic oscillator: $Z = \frac{1}{1-e^{-\hbar\omega/\tau}}$. The thermal average photon number is $< s >= \frac{1}{Z}\sum_s s\, e^{-s\hbar\omega/\tau} = \frac{1}{e^{\hbar\omega/\tau}-1}$, after some manipulation. The red curve below shows <s>, while the blue curve simply shows $< s > \sim\tau/\hbar\omega$ (which is the classical result). The two lines are parallel at large temperatures, but the photon theory shows that the thermal average photon number is exponentially suppressed at low temperatures, and this key result allowed Planck to fit the black body radiation spectrum with his newly invented quantum theory in 1900.



The thermal average energy in this single mode is $<\varepsilon>=<s\hbar\omega>=\frac{\hbar\omega}{e^{\hbar\omega/\tau}-1}$.

Now consider electromagnetic waves in a three-dimensional empty cube of side L with metallic walls. Solutions to Maxwell's equations have eigen-frequencies of the form $\omega=\frac{n\pi c}{L}$, where $n=\sqrt{n_x^2+n_y^2+n_z^2}$, and $n_x=1,2,3,\ldots$, $n_y=1,2,3,\ldots$, $n_z=1,2,3,\ldots$ There are an infinite number of modes available and each of these modes can have $s=0,1,2,3,\ldots$ photons occupying it. We have already calculated the thermal average photon number in each mode above. Now calculate the total energy of photons in the box by adding up the energy in each of the infinite number of modes: $U=\sum_n<\varepsilon_n>=\sum_n\frac{\hbar\omega_n}{e^{\hbar\omega_n/\tau}-1}$. Remember that n represents a list of three quantum numbers, so that this is a triple sum. However the summand only depends on the magnitude of n, so we can convert the triple sum to a single integral on n. However we have to count the states properly in this conversion. The modes can be described as dots in three-dimensional "$n-space$", spanned by the n_x, n_y, n_z axes. Many high-n modes will be occupied, so that the $n-space$ will be so dense with points that we can treat it as a continuous medium. All the points with nearly the same value of n will lie on or very near the surface of an octant. As this spherical surface expands out by a distance dn, it will include $\frac{1}{8}(4\pi n^2)dn$ more points (because each point takes up a volume of 1 in $n-space$). Thus the triple sum becomes $U=2\frac{4\pi}{8}\int_0^\infty\frac{\hbar\omega_n}{e^{\hbar\omega_n/\tau}-1}n^2dn$, where the factor of 2 comes from the two independent polarization states that each photon can have. The result of the integral is $U/V=\frac{\pi^2\tau^4}{15\hbar^3c^3}$, where $V=L^3$ is the volume of the box. The dependence of the energy density of the photon gas on the fourth power of temperature of the reservoir is the Stefan-Boltzmann law.

Looking at the integrand for U/V above, we can find how the energy density $u(\omega)$ is distributed over frequency. This is the famous Planck blackbody radiation law: $u(\omega)=\frac{\hbar\omega^3/\pi^2c^3}{e^{\hbar\omega/\tau}-1}$. The numerator of this expression is basically the classical prediction, and leads to the "ultraviolet catastrophe" in the limit of large frequency. The exponential in the denominator suppresses the energy density at high frequency (due to the quantized nature of the photon modes discussed above), avoids the catastrophe, and gives excellent agreement with experimental data on blackbody radiators, including the cosmic microwave background radiation spectrum.